Department of Computer & Information Sciences
Pakistan Institute of Engineering and Applied Sciences

Closure Properties of Context-Free Languages

Umar Faiz

http://www.pieas.edu.pk/umarfaiz/cis317

CFLs Closed Under Union

Context-free languages are closed under Union

L_1 is context free
L_2 is context free
⇒ $L_1 \cup L_2$ is context-free

CFLs Closed Under Union

Proof-by-construction:

There is a CFG G_{AUB} that recognizes $A \cup B$.

Since A and B are CFLs, there are CFGs $G_A = (V_A, T_A, P_A, S_A)$ and $G_B = (V_B, T_B, P_B, S_B)$ that generate A and B.

$$G_{AUB} = (V_A \cup V_B, T_A \cup T_B, P_{AUB}, S_0)$$

$$P_{AUB} = P_A \cup P_B \cup \{ S_0 \rightarrow S_A \} \cup \{ S_0 \rightarrow S_B \}$$

Assumes V_A and V_B are disjoint

CFLs Closed Under Union

In general:

For context-free languages L_1, L_2
with context-free grammars G_1, G_2
and start variables S_1, S_2

The grammar of the union $L_1 \cup L_2$
has new start variable S
and additional production $S \rightarrow S_1 \mid S_2$

CFLs Closed Under Union

Language	Grammar
$L_1 = \{a^n b^n\}$	$S_1 \rightarrow aS_1b \mid \lambda$
$L_2 = \{ww^R\}$	$S_2 \rightarrow aS_2a \mid bS_2b \mid \lambda$

Union

$L = \{a^n b^n\} \cup \{ww^R\}$ $\quad S \rightarrow S_1 \mid S_2$

CFLs Closed Under Concatenation

Context-free languages are closed under Concatenation

L_1 is context free
L_2 is context free
⇒ $L_1 L_2$ is context-free

CFLs Closed Under Concatenation

For context-free languages $L_1,\ L_2$
with context-free grammars $G_1,\ G_2$
and start variables $S_1,\ S_2$

The grammar of the concatenation L_1L_2
has new start variable S
and additional production $S \to S_1S_2$

CFLs Closed Under Concatenation

Language	Grammar
$L_1 = \{a^nb^n\}$	$S_1 \to aS_1b \mid \lambda$
$L_2 = \{ww^R\}$	$S_2 \to aS_2a \mid bS_2b \mid \lambda$

Concatenation

$L = \{a^nb^n\}\{ww^R\}$ $\qquad S \to S_1S_2$

CFLs Closed Under Star-Operation

Context-free languages are closed under Star-Operation:

L is context free ⇒ L^* is context-free

CFLs Closed Under Star-Operation

Proof-by-construction:

There is a CFG G^* that recognizes A^*.

$G = (V, \mathrm{T}, P, S)$

$G^* = (V \cup \{S_0\}, \Sigma, P^*, S_0)$

$P^* = P \cup \{ S_0 \to S \} \cup \{ S_0 \to S_0S_0 \} \cup \{ S_0 \to \varepsilon \}$

CFLs Closed Under Star-Operation

Language	Grammar
$L = \{a^nb^n\}$	$S \to aSb \mid \lambda$

Star Operation

$L = \{a^nb^n\}^*$ $\qquad S_1 \to SS_1 \mid \lambda$

CFLs Closed Under Reverse

Proof-by-construction:

There is a CFG G^R that recognizes A^R.

$G = (V, \mathrm{T}, P, S)$

$G^R = (V, \mathrm{T}, P^R, S)$

$P^R = \{ A \to \alpha^R \mid A \to \alpha \in P \}$

Negative Properties of Context-Free Languages

CFLs Closure Under Intersection

Context-free languages are not closed under intersection

L_1 is context free
L_2 is context free

$\Rightarrow$ $L_1 \cap L_2$ not necessarily context-free

CFLs Closure Under Intersection

$L_1 = \{a^n b^n c^m\}$

Context-free:

$S \rightarrow AC$

$A \rightarrow aAb \mid \lambda$

$C \rightarrow cC \mid \lambda$

$L_2 = \{a^n b^m c^m\}$

Context-free:

$S \rightarrow AB$

$A \rightarrow aA \mid \lambda$

$B \rightarrow bBc \mid \lambda$

Intersection

$L_1 \cap L_2 = \{a^n b^n c^n\}$ NOT context-free

CFLs Closure Under Complement

Context-free languages are **not** closed under complement

L is context free $\Rightarrow$ $\overline{L}$ not necessarily context-free

CFLs Closure Under Complement

$L_1 = \{a^n b^n c^m\}$

Context-free:

$S \rightarrow AC$

$A \rightarrow aAb \mid \lambda$

$C \rightarrow cC \mid \lambda$

$L_2 = \{a^n b^m c^m\}$

Context-free:

$S \rightarrow AB$

$A \rightarrow aA \mid \lambda$

$B \rightarrow bBc \mid \lambda$

Complement

$$\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2 = \{a^n b^n c^n\}$$

NOT context-free

Intersection of Context-free languages and Regular Languages

Intersection of a CFL and RL

The intersection of a context-free language and a regular language is a context-free language

L_1 context free

L_2 regular

$\Rightarrow$ $L_1 \cap L_2$ context-free

Intersection of a CFL and RL

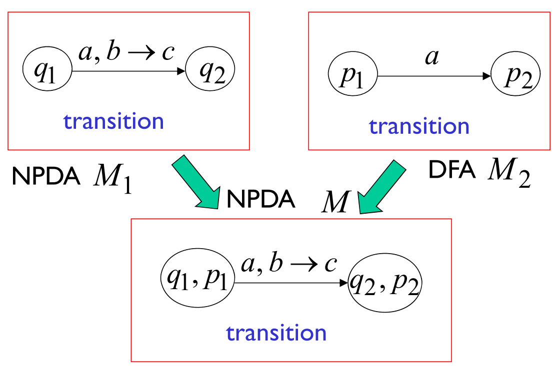
NPDA for L_1 context-free — Machine M_1

DFA for L_2 regular — Machine M_2

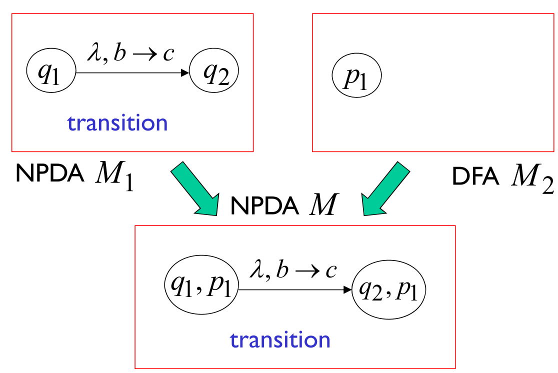
Construct a new NPDA machine M that accepts $L_1 \cap L_2$

M simulates in parallel M_1 and M_2

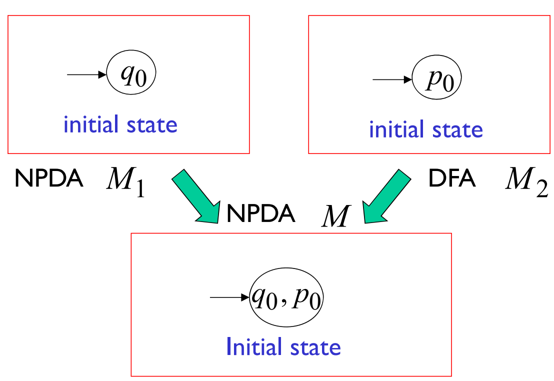
Intersection of a CFL and RL



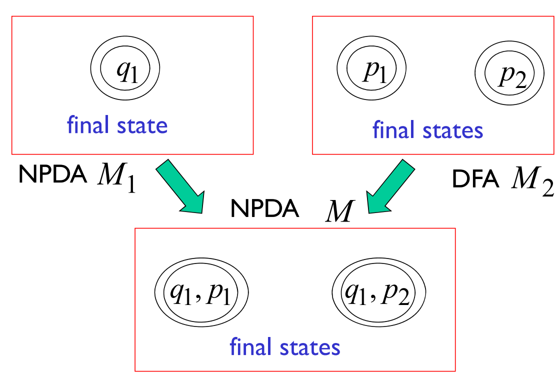
Intersection of a CFL and RL



Intersection of a CFL and RL



Intersection of a CFL and RL



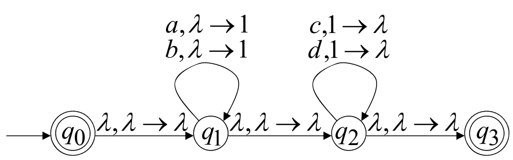
Intersection of a CFL and RL

Example:

$$L_1 = \{w_1 w_2 : |w_1| = |w_2|, w_1 \in \{a,b\}^*, w_2 \in \{c,d\}^*\}$$

context-free

NPDA M_1

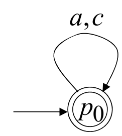


Intersection of a CFL and RL

$$L_2 = \{a,c\}^*$$

regular

DFA M_2

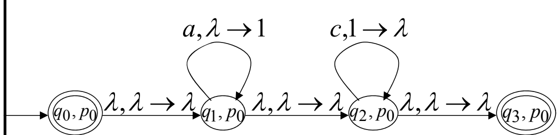


Intersection of a CFL and RL

Automaton for: $L_1 \cap L_2 = \{a^n c^n : n \geq 0\}$

context-free

NPDA M



Intersection of a CFL and RL

In General:

M simulates in parallel M_1 and M_2

M accepts string w if and only if

M_1 accepts string w and

M_2 accepts string w

$$L(M) = L(M_1) \cap L(M_2)$$

Intersection of a CFL and RL

Therefore:

M is NPDA

↓

$L(M_1) \cap L(M_2)$ is context-free

↓

$L_1 \cap L_2$ is context-free

Decidable Properties of Context-Free Languages

Decidable Properties of CFLs

Membership Question

For context-free grammar G find if string $w \in L(G)$

Membership Algorithms

Parsers

- Exhaustive search parser
- CYK parsing algorithm

Decidable Properties of CFLs

Empty Language Question

For context-free grammar G, find if $L(G) = \emptyset$

Algorithm

1. Remove useless variables
2. Check if start variable S is useless

Decidable Properties of CFLs

Infinite Language Question

For context-free grammar G, find if $L(G)$ is infinite

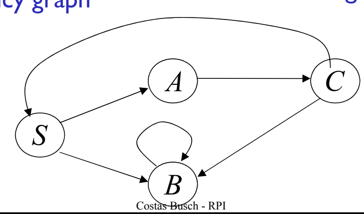
Algorithm

1. Remove useless variables
2. Remove unit and λ productions
3. Create dependency graph for variables
4. If there is a loop in the dependency graph, then the language is infinite

Example:
$$S \to AB$$
$$A \to aCb \mid a$$
$$B \to bB \mid bb$$
$$C \to cBS$$

Dependency graph

Infinite language



$$S \to AB$$
$$A \to aCb \mid a$$
$$B \to bB \mid bb$$
$$C \to cBS$$

$$S \Rightarrow AB \Rightarrow aCbB \Rightarrow acBSbB \Rightarrow acbbSbbb$$

$$S \overset{*}{\Rightarrow} acbbSbbb \overset{*}{\Rightarrow} (acbb)^2 S (bbb)^2$$

$$\overset{*}{\Rightarrow} (acbb)^i S (bbb)^i$$